Quantum networks theory
— *Internship proposal*

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Based on the Quantum journal paper *quantum networks theory*, see also the video abstract or the talk.

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Context. In classical Computer Science many composite systems are modelled by dynamical networks, for instance computer processes, neurons, biochemical agents, particle systems, market agents and social network users. This because those systems, e.g. social networks agents, have the capabilities to spawn, disappear, connect and disconnect. Whilst standard quantum theory focusses on the quantisation of the individual systems within networks, a recently developed quantum networks theory seeks to quantise *all features* of dynamical networks—including their connectivity and population (see Fig. 1).

This echoes a fundamental question in Computer Science, which is simply : What is a computer? Can we come up with a mathematical definition that captures the fundamental resources that are granted to us by nature, namely both spatial parallelism and quantum parallelism? Both of them appear in the quantum circuit model of computation, but independently. Can we develop a model of quantum computing in which spatial parallelism itself can be made subject to quantum parallelism? A 'fully quantum internet'? Even the quantisation of connectivity alone within distributed quantum computers [3, 4] can be used to implement protocols [13, 17, 18] in which the orderings of events [7] and trajectories of particles [8] are quantum in their specification, leading to communitation complexity [8, 11, 16] and algorithmic complexity advantages [1, 2, 12].

As it turns out, motivations for quantising networks also appear in the foundations of physics. Whilst a theory which satisfactorily quantises gravity remains elusive, a common feature of most attempts is the superposition of spacetimes geometries, infact it is this feature [9, 10] which expected to be testable by near-future experiments [5, 6, 14, 15].

The above considerations led the Arrighi et al. to construct a quantum networks theory, which from the very start is equipped with robust and well-behaved notions of unitarity, locality, and

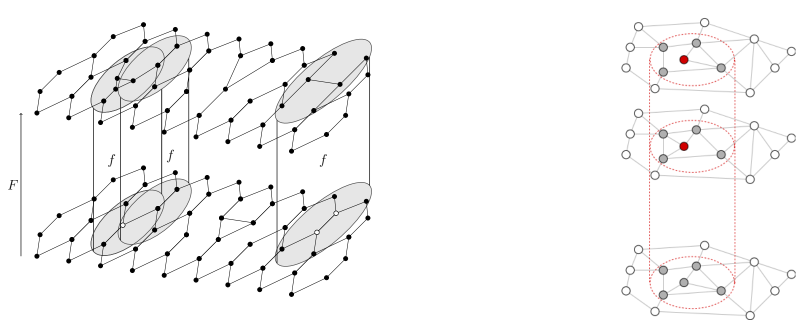

FIG. 1. *Classical and a quantum network dynamics.*

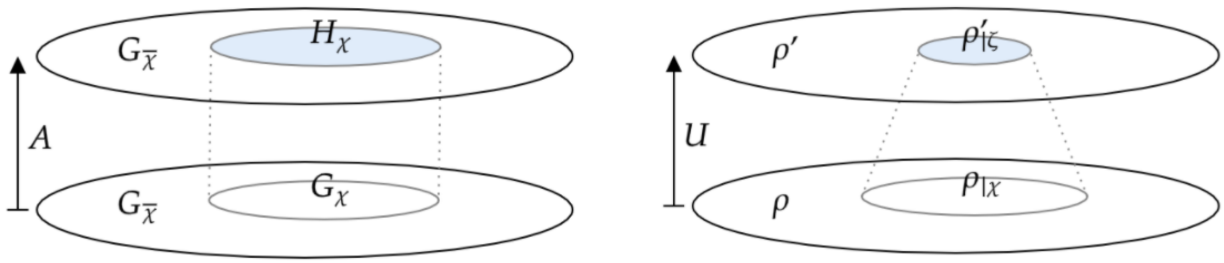

FIG. 2. *Local operators.* **Left**: An operator A is χ-local if it only modifies G_χ. **Right**: An operator U is $\chi\zeta$-causal if it will only allow for the $|\zeta$ part of its output to depend on the $|\chi$ part of its input

causality (see Fig. 2).

Mathematical overview. In this quantum networks theory, classical network configurations are quantised by using them to freely generate a Hilbert space. This allows for arbitrary quantum superpositions of networks, e.g. having different connectivities and node populations. As a result, the standard tensor product becomes inappropriate for modelling of parallel composition and the partial trace becomes inappropriate for modelling the reduction of a state to a subsystem. These primitive operations of discrete quantum theory need be carefully generalized, based on the notion of a *restriction*.

A restriction is a recipe for selecting a subnetwork within a network. Formally modelled by a function $\chi : \mathcal{G} \to \mathcal{G}$ on the space $\mathcal{G}$ of all networks, its action will be denoted by $\chi : G \mapsto G_\chi$, with $G_\chi \subseteq G$. Restrictions can be seen as a way of partitioning the world into two parts, taking each possible network and splitting it into G_χ and the remainder $G_{\bar{\chi}}$. They are extremely flexible, for instance they can be used to select the nodes that have certain names, or those that hold a certain state, they are stable under taking unions, compositions and even neighbourhoods in the network, whether directed or undirected. A given χ really just defines 'what systems are closeby' for a specific purpose, leaving this general and modular.

A first series of results shows that every restriction induces a *partial trace*

$$(|G\rangle\langle H|)_{|\chi} := |G_\chi\rangle\langle H_\chi|\langle H_{\overline{\chi}}|G_{\overline{\chi}}\rangle$$

which is well-behaved over arbitrary quantum networks (trace-preserving, completely positive).

A second series of results shows that every restriction also leads to a tensor product defined by

$$|G_\chi\rangle \odot |G_{\overline{\chi}}\rangle := |G\rangle$$

and $|H\rangle \odot |K\rangle := 0$ whenever H and J are not χ-consistent, that is when they do not result from the application of χ and $\overline{\chi}$ to some G. The operation $\odot$ and the corresponding notion of *consistency* can be lifted to density matrices and operators, where usual intuitions about the standard tensor product $A \otimes B$ typically carry through to $A \odot B$, provided that χ-consistency conditions are met.

An operator is considered to be local on a restricted part χ of a network if it alters only that which is within χ, ignoring the remainder. I.e. χ-locality of A is defined as $\langle H| A |G\rangle = \langle H_\chi| A |G_\chi\rangle \langle H_{\overline{\chi}}|G_{\overline{\chi}}\rangle$, requiring that the amplitude for transition from G to H depends only on the action of A on the subgraphs G_χ and H_χ. This locality principle is defined directly in terms of restriction, but remarkably it can be equivalently defined with respect to the newly developed notions of parallel composition and reduction to a subsystem:

$$A \text{ is } \chi\text{-local} \iff A = A \otimes I \iff \forall \rho,\ (A\rho)_{|\varnothing} = (A\rho_{|\chi})_{|\varnothing}$$

An operator U is considered to be $\chi\zeta$-causal if the ζ-part of the output of U depends entirely on the χ-part of the input of U. This intuition is captured formally by requiring that U should satisfy $\left(U\rho U^\dagger\right)_{|\zeta} = \left(U\rho_{|\chi}U^\dagger\right)_{|\zeta}$. This definition in terms of generalized partial trace, is furthermore well-behaved with respect to locality and parallel composition, e.g. we have

$$U \text{ is } \chi\zeta \text{ causal} \iff \forall A\ \chi\text{-local} : UAU^\dagger \text{ is } \zeta\text{-local}$$

The well-behaviour of locality and causality with respect to generalized tensors and traceouts, is not only suggestive of the appropriateness of their redefinition in terms of restrictions, but in fact sufficient to provide a powerfultool box for reasoning and proving theorems about locality and causality in fully quantum theories of network dynamics.

Questions. Quantum networks theory is recent and offers many opportunities for improvements, e.g. worked out examples; providing a version with explicit edges; developping its open quantum systems theory...

It also provides a new toolbox in order to tackle fundamental theorems e.g. on the equivalence between causality constraints (e.g. formulated in terms of non-signalling or commuting algebras) and the more hands-on circuits of local gates constructions; on entanglement between internal degrees of freedom and how this may relate to contextuality; on modelling delocalized observers and how this relates to the topical subject of quantum reference frames...

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